Massachusetts Institute of Technology
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LUMINARY Memo # 63

TO: Distribution

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SUBJECT: A Derivation of the Improved Lunar Landing Guidance Equations

Introduction

LUMINARY Memo #62 exhibited a set of lunar landing guidance equations which compensate for computation, throttle, and attitude control lags by projecting the guidance commands forward from the state vector time. This memo gives a straight forward derivation of these equations. It also casts them in two different forms in order to emphasize a trade-off between computational simplicity and speed (very desirable) and certain dynamical advantages. The simpler equations result from the combination and cancelling of terms, yielding a computer execution time savings but a loss of certain dynamical advantages. However, the main advantage, the projection of the guidance command forward, is retained. Re-arrangement of this simplified equation, Eq. (15) below, yields exactly the same formulation Bill Widnall and Allan Klumpp have independently derived. Naturally, all the solutions are equivalent since the quadratic acceleration solution to the lunar landing two point boundary valued problem is unique.

The more complicated form of the equations are expressed in a way which permits all the following:

1. Projection of the throttle command forward τ_1 seconds and the attitude command forward τ_2 seconds in order to account for the different time constants in the throttle response and the DAP response.

2. Expression of the solution acceleration regime in terms which can be projected foward to any time without measuring the state

near that time or recomputing new parameters. It is hard to explain this advantage before the derivation itself is explained.

3. Expression of the equations in a form which allows "freezing" the position constraint parameter independently of the velocity constraint parameter. (There is a distinct difference between "freezing" the position constraint parameter and abandoning position control. When the position, or velocity, constraint is frozen, position, or velocity, control is not abandoned; but new state information is not processed.)

4. The solution thrust orientation profile is not, of course, a constant direction in space. The solution thrust vector turns in inertial space. The present interface between the guidance equations and the attitude control system do not take this desired turning rate analytically and explicitly into account. The more complex form of the equations below show how this desired turning rate can be exactly computed.

This memo has three main purposes:

i. A tutorial one: to let interested people understand the basic guidance equations and their modification to combat the attitude oscillation problem referred to in LUMINARY Memo #62 and Apollo Project Memo #7-69.

ii. To show that there is a tradeoff in the expression of the guidance equations between a formulation retaining the four advantages above and the very important advantage of reduced computer execution time.

iii. To prepare the groundwork for a Program Change Notice to change the equations in the GSOP and the LUMINARY 1A program.

Derivation

The definitions given in LUMINARY Memo #62 are applicable here also.

The lunar landing guidance equations are frequently called a quadratic guidance law. The term "quadratic" refers to the fact that the solution total acceleration components are expressed as quadratic functions of time. Expressed as a vector, then, the total acceleration will have the following form.

$$\ddot{\underline{r}}(t) = \underline{a} + (T - t)\underline{b} + (T - t)^2\underline{c} \tag{1}$$

To derive the quidance law, we simply have to determine $\underline{a}$, $\underline{b}$ and $\underline{c}$ in such a way that the following five equations are satisfied.

$$\ddot{\underline{r}}(T) = \underline{r}_D = \underline{a}_{TD}(T) + \underline{g}(T) \tag{2}$$

$$\underline{v}(T) = \underline{v}_D \tag{3}$$

$$\underline{r}(T) = \underline{r}_D \tag{4}$$

$$v(T) = \underline{v}_o + \int_{t_o}^{T} \ddot{\underline{r}}\,(t)\,dt \tag{5}$$

$$\underline{r}(T) = \underline{r}_o + T_{go}\underline{v}_o + \int_{t_o}^{T} (T - t)\,\ddot{\underline{r}}(t)\,dt \tag{6}$$

Equations (5) and (6) are integrals of the equations of motion; they relate the state at $t = t_o$ and the total kinematic acceleration profile $\ddot{\underline{r}}(t)$ between t_o and T, to the terminal state at $t = T$. They are often called the final value equations. (The factor multiplying the total acceleration under the integral sign in Eq. (6) is the influence function of acceleration on terminal position.)

The strategy now is straight forward: generate three vector equations in the three unknowns $\underline{a}$, $\underline{b}$ and $\underline{c}$ by

1. substituting Eq. (2) into Eq. (1) in order to satisfy the final desired thrust vector constraint and

2. substituting Eqs. (1) and (3) into Eq. (5) in order to satisfy the final desired velocity vector constraint and

3. substituting Eqs. (1) and (4) into Eq. (6) in order to satisfy the final desired position vector constraint.

In performing the integrations of Eq. (1) and its product with (T - t), it is necessary only to know the following formula

$$\int_{t_o}^{T} (T - t)^n \, dt = T_{go}^{n+1} / (n + 1) \tag{6}$$

The results of the three substitutions are, respectively,

$$\underline{\ddot{r}}_D = \underline{a} \tag{7}$$

$$\underline{v}_D - \underline{v}_o = T_{go}\underline{a} + (T_{go}^{2}/2)\,\underline{b} + (T_{go}^{3}/3)\,\underline{c} \tag{8}$$

$$\underline{r}_D - (\underline{r}_o + T_{go}\underline{v}_o) = (T_{go}^{2}/2)\underline{a} + (T_{go}^{3}/3)\underline{b} + (T_{go}^{4}/4)\underline{c} \tag{9}$$

Equation (7) determines $\underline{a}$. We are going to determine $\underline{b}$ in such a way that given any $\underline{a}$ and $\underline{c}$, the velocity constraint equation is satisfied. This will lead to advantage 3. referred to in the introduction. Solving Eq. (8) for $\underline{b}$ yields

$$\underline{b} = (2/T_{go}^{2})(\underline{v}_D - \underline{v}_o) - (2/T_{go})\,\underline{\ddot{r}}_D - (2T_{go}/3)\,\underline{c} \tag{10}$$

Given <u>any</u> $\underline{\ddot{r}}_D$ and $\underline{c}$, determination of $\underline{b}$ from Eq. (10) and commanding thrust acceleration satisfying Eq. (1) will result in satisfaction of the velocity constraint problem.

Now $\underline{c}$ must be determined to satisfy Eq. (9), the position constraint problem. This is done by substituting Eq. (7) and Eq. (10) into Eq. (9) and

then solving for $\underline{c}$. The result of this operation is

$$\underline{c} = (36/T_{go}^4)[\underline{r}_D - (\underline{r}_o + T_{go}\underline{v}_o)] - (24/T_{go}^3)(\underline{v}_D - \underline{v}_o) + (6/T_{go}^2)\ddot{\underline{r}}_D \quad (11)$$

Note the first term in Eq. (11); it is the product of two factors. The first factor, which has the fourth power of time-to-go in the denominator, is a kind of gain factor; it multiplies the final position error which would result if the acceleration were turned completely off. Because the influence function for final position (see Eq. (6)) is (T - t), the effectiveness of acceleration to control final position is infinitesimally small at $T - \epsilon$. Therefore, the gain multiplying any predicted final errors must approach infinity very rapidly as time-to-go approaches zero. For this reason, unrelinquishing position control, for practical reasons, is usually dropped before the velocity control. This can conveniently be done by "freezing" $\underline{c}$ (not recomputing it from Eq. (11)) and continuing to compute $\underline{b}$ from Eq. (10).

Expression of the Equations in Simple Form

Before we cast the equations in the form which maintains the advantages referred to in the introduction, we will cast them in the form which emphasizes computer storage and execution time economies.

Remember, we want to project the guidance commands forward from t_o, the state vector time, to compensate for computation and control system lags. So we advance the command acceleration forward τ seconds to t^* where

$$t^* = t_o + \tau \quad (12)$$

$$T_{go}^* = T_{go} - \tau \quad (13)$$

Then Eq. (1) becomes

$$\ddot{\underline{r}}_c(t^*) = \underline{a} + T_{go}^*\underline{b} + (T_{go}^*)^2\underline{c} \quad (14)$$

Substituting the expressions for $\underline{a}$, $\underline{b}$ and $\underline{c}$ (which are found in Eqs. (7), (10), and (11)) into Eq. (14) and combining terms yields the following simplified guidance law.

$$\ddot{\underline{r}}_c(t^*) = R(3R - 2)\,(12/T_{go}^2)\,[\underline{r}_D - (\underline{r}_o + T_{go}\underline{v}_o)] - R(4R - 3)\,(6/T_{go})\,(\underline{v}_D - \underline{v}_o) + [1 + 6R(R - 1)]\,\ddot{\underline{r}}_D \tag{15}$$

where

$$R = T_{go}^* / T_{go} \tag{16}$$

the ratio of the diminished time-to-go to the time-to-go from state vector time. Note that if guidance command projection is not used the command acceleration is simply

$$\ddot{\underline{r}}_c(t_o) = (12/T_{go}^2)\,[\underline{r}_D - (\underline{r}_o + T_{go}\underline{v}_o)] - (6/T_{go})\,(\underline{v}_D - \underline{v}_o) + \ddot{\underline{r}}_D \tag{17}$$

There fore the effect of the guidance command projection is to multiply each normal gain factor on the predicted position and velocity error by another factor. The behavior of these factors is very interesting. The factor for position is sketched in Figure 1; i.e., the function $R(3R - 2)$ is plotted versus R. Note the change of sign when R is less than 2/3! (The factor for velocity is also a parabola which is concave upward but its roots are at $R = 0$ and $R = 3/4$; its minimum occurs at (3/8, 9/16).) The change in sign is due to the predicted flip in sign of the predicted final error. As an example, if $\tau = 3$, the sign change at $R = 2/3$ will occur at $T_{go} = 9$. This means that at six seconds before T the sign of the predicted final position error will change. The factor $R(3R - 2)$ predicts or projects this sign change.

Equation (15) is valid until $R = 0$; i.e., until the diminished time-to-go is zero. These equations are nearly in the form derived by Allan Klumpp and Bill Widnall.

Expression of the Equations in a Dynamically Superior But Computationally More Complex Form

We will now express the equations in a form which preserves the four additional advantages referred to in the introduction. All the program steps are suggested here:

1. Determine $\underline{r}_o$, $\underline{v}_o$, t_o and T_{go}. That is, read the computer clock, the PIPAS, perform the navigation equations and calculate T_{go}.

2. Compute the guidance parameter $\underline{c}$ if T_{go} is greater than, say, ten seconds; otherwise, use the last computed value of $\underline{c}$

$$\underline{c} = \text{Eq. (11) if } T_{go} > 10$$

$$\underline{c} = \text{last computed } \underline{c} \text{ if } T_{go} \leq 10$$

3. Compute the guidance parameter $\underline{b}$

$$\underline{b} = \text{Eq. (10)}$$

4. Compute the projected throttle command

$$a_{TD}(t_1^*) = \text{abval}\,[\underline{a} + (T_{go} - \tau_1)b + (T_{go} - \tau_1)^2\,\underline{c} - \underline{g}]$$

5. Compute the projected attitude control system command

$$\underline{\text{axis}}_D(t_2^*) = \text{unit}\,[\underline{a} + (T_{go} - \tau_2)\,\underline{b} + (T_{go} - \tau_2)^2\,\underline{c} - \underline{g}_o]$$

6. Compute the desired thrust attitude turning rate

$$\underline{\omega}_D = [\underline{\text{axis}}_D(t_2^*) * \underline{a}_{TD}(t_2^*)] / a_{TD}(t_1^*)$$

where

$$\underline{a}_{TD}(t_2^*) = -\underline{b} - 2(T_{go} - \tau_2)\,\underline{c} - \underline{g}$$

Note that at no time is it necessary to change the form of the equations - to change to linear guidance or velocity nulling guidance, etc. The parameters $\underline{a}$, $\underline{b}$ and $\underline{c}$ can be used in steps 4 through 6 right up to the time when the diminished time-to-go equals zero even if steps 2 and 3 are skipped. Furthermore, these parameters will solve the boundary-valued problem as it is understood at the last computation of $\underline{a}$, $\underline{b}$ and $\underline{c}$.

Summary

Two forms of the lead lunar landing guidance equations have been derived - a complicated form which permits certain advantages and a simple form whose simplicity speaks for itself.

If the main or only advantage of the complicated form is projecting by a different time for throttle and the DAP, then the simpler form is still to be preferred since Eq. (15) can be computed twice (saving from the first computation, of course, the common factors) with a different R each time. The use of signal $\underline{\omega}_D$ with the present guidance/control interface is anything but straight forward.

Conclusion

A PCN should be generated to mechanize Eq. (15) if all the analysis and simulations verify the hypothesis that this will materially reduce the attitude oscillation problem without introducing a new problem. At some value of T_{go} (TBD), the guidance law can be changed from Eq. (15) to a simple velocity-nulling law.

$$\underline{a}_{TC} = (\underline{v}_D - \underline{v}_o)/\tau$$

thereby relaxing the positional constraint when it becomes impractical to maintain it. This modification, then, represents a minimum change from the current program design.

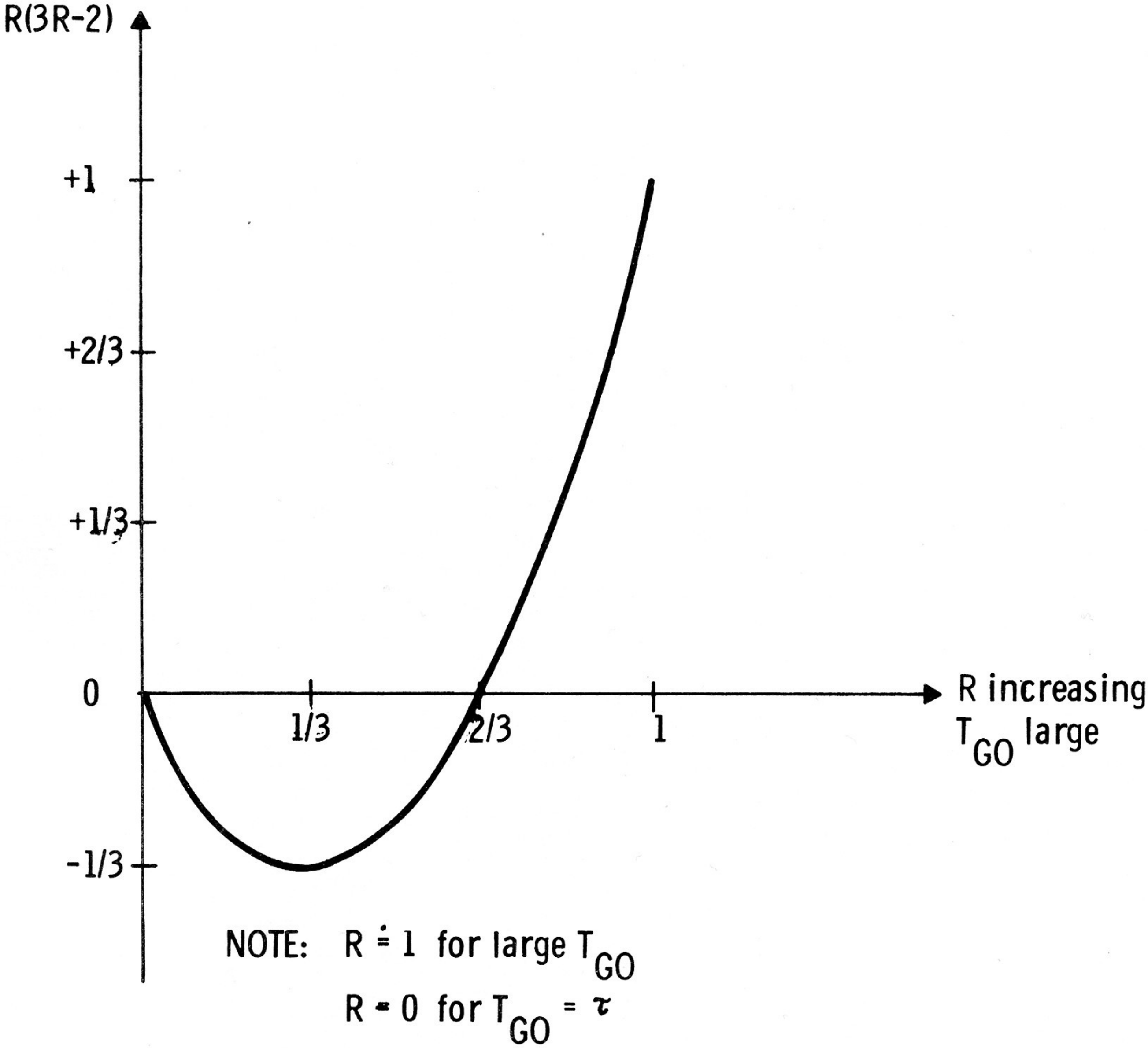

Fig. 1 Plot of Position Constraint Projection Factor.